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An exact solution of extended Graetz problem with axial heat conduction

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Abstract—The convective heat transfer properties of a hydrodynamically, fully developed viscous flow in a circular tube are analyzed without using any simplified assumption such as low Reynolds numbers or Peclet numbers. The pipe is then subjected to a step-change in wall temperature. A straightforward approach using the Fourier transform technique is utilized to obtain analytical expressions for temperature distribution, heat flux, and Nusselt numbers. The effects of axial heat conduction are included in these expressions in both the upstream and the downstream directions for Peclet numbers ranging from 0 to ∞. By first taking the Fourier transform of the temperature field and expanding the coefficients of the transformed temperature in terms of the Peclet number, the energy equation with discontinuous wall temperature and longitudinal heat conduction is transformed into a set of ordinary differential equations. This resulting set of equations is then solved successively. The representative curves illustrating the variations of bulk temperature, heat flux, and Nusselt numbers with pertinent parameters are plotted. The asymptotic Nusselt numbers for small, as well as large Peclet numbers, is obtained as 3.6565, compared to the exact classical value of 3.6568. The significance of each curve is also discussed.

1. INTRODUCTION

THE EXTENDED Graetz problem is considered with the effect of longitudinal heat conduction in the entire flow field for Peclet numbers ranging from 0 to ∞. Diffusive and convective transport of energy in this kind of problem is governed by the magnitude of the Peclet number. The smaller the Peclet number, the greater the heat transfer by conduction in the flow direction. By this effect, the temperature not only penetrates in the flow downstream direction, but also in the flow upstream direction. This requires an upstream wall boundary condition. In fact, an extensive survey of existing literature on the Graetz problem is presented in refs. [1–6]. This survey indicates that the Graetz problem has not yet been investigated in complete generality. One natural and practical boundary condition involves a uniform upstream wall temperature condition. This case has not been especially studied in the literature. The solutions presented in refs. [7–10] are a two region analysis and a matching technique. In addition, an insulated upstream wall temperature is considered, which is different from the boundary condition considered in this paper. The solution presented in ref. [11] is a numerical technique using a double-sided Laplace transform with inversion to obtain a solution for the downstream region of the flow which is different from the solution of the present work. Reference [12] has presented numerical solutions with axial conduction for insulated constant temperature and uniform heat flux case. By trial and error, the author approximated the boundary condition at infinity. Reference [13] has developed an exact solution to the uniform-wall-heat-flux case by a separation of variables technique. In refs. [14, 15], the solution is numerical, and again, an insulated boundary condition is used. The results presented in ref. [16] are a series of product solutions to the differential equation subjected to a constant heat flux boundary condition, which is, again, different from the boundary condition considered in this paper. The most recent work for the extended Graetz problem is given in refs. [17, 18]. However, the analysis does not contain the calculation of the wall heat transfer coefficients (Nusselt numbers). For this reason, technical conclusions cannot be obtained and comparison is not possible.

The purpose of this paper is to present an analytical solution to the Graetz type problem with axial heat conduction for a fully developed Poiseuille flow in a circular tube subjected to a step increase in wall temperature. The method of solution is by the Fourier integral transformation technique.

It is believed that availability of such a solution to the designer and practitioner will be an important contribution to the design and analysis of heat exchangers as evaporator and condensers, and in fact any heat exchanger having headers with protruding piping connections.

2. SOLUTION OF THE ENERGY EQUATION

The problem considered is that of an infinite tube of circular cross-section through which a fluid is flowing in a steady Poiseuille flow. One semi-infinite half of the tube is maintained at a uniform temperature, $-T_1/2$, and the other half of the tube is maintained at

NOMENCLATURE

A	coefficient of Fourier integral, equation (16)
a_1	function defined by equation (31)
a_2	function defined by equation (35)
a_3	function defined by equation (39)
B	function defined by equation (15)
$\bar{B}$	coefficient of Fourier integral, equation (16)
b_0	function defined by equation (26)
b_1	function defined by equation (29)
b_2	function defined by equation (33)
b_3	function defined by equation (37)
C	factor defined by equation (25)
C_0, C_2, C_4, C_6	functions defined by equation (A1)
$\bar{C}_0, \bar{C}_2, \bar{C}_4, \bar{C}_6$	functions defined by equation (A4)
$\bar{\bar{C}}_0$	function defined by equation (A8)
$\bar{\bar{C}}_2$	function defined by equation (A9)
$\bar{\bar{C}}_4$	function defined by equation (A10)
$\bar{\bar{C}}_6$	function defined by equation (A11)
C_p	specific heat at constant pressure, equation (3)
D_1, D_2, D_4	functions defined by equation (A5)
$\bar{D}_1, \bar{D}_2, \bar{D}_4$	functions defined by equation (A7)
D_h	hydraulic diameter, equation (49)
E_0, E_2, E_4, E_6	functions defined by equation (A6)
E_m	mixed mean excess temperature, equation (44) [K]
F	characteristic heat flux
h	convective heat transfer coefficient, equation (48) [W m^{-2} K^{-1}]
I_{00}	factor defined by equation (45)
$\bar{I}_0, \bar{I}_2, \bar{I}_4, \bar{I}_6$	functions defined by equation (A12)
$I_0(\rho)$	function defined by equation (A3)
I_0	modified Bessel function of zeroth order
I_1	modified Bessel function of the first order
J	function defined by equation (45)
k	thermal conductivity [W m^{-1} K^{-1}]
L	characteristic length [m]
m	function defined by equation (A2)
Nu	Nusselt number, equation (48)
P	circumferences of peripheries, equation (47)
Pe	Peclet number, equation (6)
Q	mass flow rates of fluid [kg s^{-1}]
R	radius of conduit [m]
r	radial coordinate
Re	Reynolds number, wL/v
S, ds	full and elemental cross-sectional areas of pipe [m^2]
T_{wall}	wall temperature [K]
T_m	mixed mean temperature, equation (43) [K]
T, t	dimensional and dimensionless temperature
T_1	sudden wall temperature change [K]
U	dimensionless heat flux from the wall, equation (40)
w, w_0	velocity and maximum velocity components, respectively
$\bar{w}$	average velocity, equation (2)
x	longitudinal coordinate.

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
η	dimensionless radial coordinate, r/R, equation (4)
λ	function defined by equation (12)
ξ	dimensionless longitudinal coordinate, x/R, equation (4)
ρ	density of fluid [kg m^{-3}].

a uniform temperature of $T_1/2$. In this paper, as is always the case, the physical parameters, thermal diffusivity, density, etc., are assumed to be independent of temperature. The geometry of the problem is defined in Fig. 1.

The energy equation for which the flow is hydrodynamically fully developed, without internal heat generation or heat sink, and is subjected to a step-change in wall temperature is given as (Fig. 1)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}w\frac{\partial T}{\partial x} \quad (1)$$

where

$$w=2\bar{w}\left(1-\frac{r^2}{R^2}\right) \quad (2)$$

with

$$\bar{w}=\frac{w_0}{2}$$

and α is the thermal diffusivity given by

$$\alpha=\frac{k}{\rho C_p}. \quad (3)$$

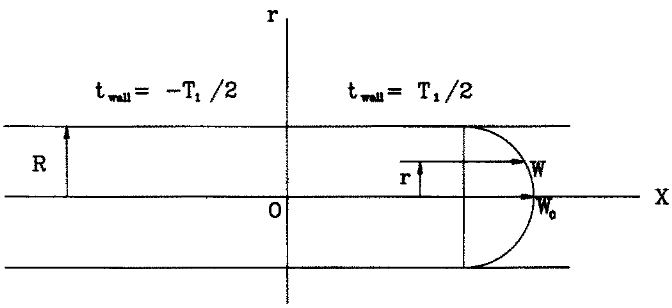
Also, T in equation (1) is the temperature distribution sought.

After introducing the dimensionless coordinates as

$$\eta=\frac{r}{R} \quad \text{and} \quad \xi=\frac{x}{R} \quad (4)$$

energy equation (1) reduces to the form

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial t}{\partial \eta}\right)+\frac{\partial^2 t}{\partial \xi^2}=Pe(1-\eta^2)\frac{\partial t}{\partial \xi}. \quad (5)$$



WHERE,
R = Radius of conduit
X, r = Longitudinal and radial coordinate, respectively
T_1 = Sudden wall temperature change at X = 0
W, W_0 = Velocity and maximum velocity components, respectively

FIG. 1. The geometry.

The factor on the right-hand side of equation (5) indicates the Peclet number, Pe, and is defined by

$$Pe = \frac{2\bar{w}R}{\alpha}. \tag{6}$$

The boundary condition in Fig. 1 is

$$t_{\text{wall}} = \begin{cases} -\dfrac{T_1}{2} & \text{for } \xi < 0 \\ \dfrac{T_1}{2} & \text{for } \xi > 0 \end{cases}. \tag{7}$$

In addition, the physical condition on the ξ-axis to be satisfied is

$$\frac{\partial t}{\partial \eta} = 0 \quad \text{for} \quad \eta = 0. \tag{8}$$

It is well known that, with the inclusion of an axial heat conduction term, energy equation (5), under the stated condition, is not amenable to the usual Sturm–Liouiville system [3, 19]. Therefore, the correct and exact solution can be obtained only by the Fourier integral transformation technique. Accordingly, the Fourier integral of energy equation (5) is

$$t(\eta, \xi) = \int_0^\infty [A(\eta\,;\alpha)\cos\alpha\xi + B(\eta\,;\alpha)\sin\alpha\xi]\,\mathrm{d}\alpha \tag{9}$$

where $-\infty < \xi < \infty$, and

$$A(\eta, \alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} t(\eta, \xi)\cos\alpha\xi\,\mathrm{d}\xi$$

$$B(\eta, \alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} t(\eta, \xi)\sin\alpha\xi\,\mathrm{d}\xi. \tag{10}$$

The substitution of equation (9) into differential equation (5), and equating the coefficients of $\cos\alpha\xi$ and $\sin\alpha\xi$ yields

$$A'' + \frac{1}{\eta}A' = \alpha^2 A + \alpha\lambda(1-\xi^2)B$$

$$B'' + \frac{1}{\eta}B' = \alpha^2 B - \alpha\lambda(1-\xi^2)A \tag{11}$$

where

$$\lambda = Pe \tag{12}$$

and the prime in equation (11) indicates differentiation with respect to the dimensionless radial coordinate η.

Before using boundary conditions (7), the wall temperature distribution must be expressed in the Fourier integral form [20, 21], as

$$\frac{1}{\pi}\int_0^\infty \frac{\sin\alpha\xi}{\alpha}\,\mathrm{d}\alpha = \begin{cases} \dfrac{1}{2} & \text{when} \quad \xi > 0 \\ 0 & \text{when} \quad \xi = 0. \\ -\dfrac{1}{2} & \text{when} \quad \xi < 0 \end{cases} \tag{13}$$

Equation (13) must be equal to the temperature function, equation (9), when evaluated at $\eta = 1$, that is

$$\eta = 1; \quad A = 0; \quad B = \frac{T}{\alpha\pi}. \tag{14}$$

The condition of B is nonhomogeneous, which can be made homogeneous if a new function $\bar{B}$ is defined by the relation

$$B = \bar{B} + \frac{T}{\alpha\pi} \tag{15}$$

for which

$$\bar{B} = 0 \quad \text{when} \quad \eta = 1.$$

The problem as posed by differential equation (5) and boundary condition (7) is a non-linear problem because of the nonlinearity that is caused by the step-change in the wall temperature. To transfer this non-linear problem into a set of linear problems, the following methods of solution are used.

The coefficients of the Fourier integral equation (9) are expanded in terms of the Peclet number as

$$\bar{B} = b_0+\lambda b_1+\lambda^2 b_2+\lambda^3 b_3+\dots \quad \text{where} \quad b_i = b_i(\eta)$$
$$A = \lambda a_1+\lambda^2 a_2+\lambda^3 a_3+\dots \text{ where } \quad a_i = a_i(\eta). \quad (16)$$

The functions, a_i's and b_i's will satisfy the following symmetry and boundary conditions:

$$\begin{aligned} \eta = 0: b_0' = 0, \quad & b_1' = 0, \quad b_2' = 0,\dots \\ & a_1' = 0, \quad a_2' = 0,\dots \\ \eta = 1: b_0 = 0, \quad & b_1 = 0, \quad b_2 = 0,\dots \\ & a_1 = 0, \quad a_2 = 0,\dots \end{aligned} \quad (17)$$

It can be seen that the boundary conditions in equation (17) are not only linear, but also homogeneous.

Substituting the expansion of $\bar{B}$ and A from equation (14) into the differential equations of A and B (after making the necessary change from B to $\bar{B}$), and collecting the equal powers of λ, the following set of ordinary linear differential equations is obtained:

$$\begin{aligned} b_0''+\frac{1}{\eta}b_1'-\alpha^2 b_0 &= \frac{\alpha T}{\pi} \\ b_1''+\frac{1}{\eta}b_1'-\alpha^2 b_1 &= 0 \\ b_2''+\frac{1}{\eta}b_2'-\alpha^2 b_2 &= -\alpha(1-\eta^2)a_1 \\ b_3''+\frac{1}{\eta}b_3'-\alpha^2 b_3 &= -\alpha(1-\eta^2)a_2 \\ &\dots\dots \\ a_1''+\frac{1}{\eta}a_1'-\alpha^2 a_1 &= \alpha(1-\eta^2)b_0+\frac{T}{\pi}(1-\eta^2) \\ a_2''+\frac{1}{\eta}a_2'-\alpha^2 a_2 &= \alpha(1-\eta^2)b_1 \\ a_3''+\frac{1}{\eta}a_3'-\alpha^2 a_3 &= \alpha(1-\eta^2)b_2. \\ &\dots\dots \end{aligned} \quad (18)$$

The above equations, starting from the equation of b_0, can be solved successively under boundary condition (17). Note that the function b_0 represents a temperature distribution for a zero Peclet number which is a conduction heat transfer in a rod subjected to a step-change in wall temperature. The solutions of the differential equations given in equation (18) are outlined below.

2.1. *Solution of the differential equation for* b_0

The differential equation for b_0 can be written from the first equation of (18) as

$$b_0''+\frac{1}{\eta}b_0'-\alpha^2 b_0 = \frac{\alpha T}{\pi} \quad (19)$$

where the primes indicate differentiation with respect to the dimensionless radial coordinate η. If the independent variable η is changed to ρ by the transformation

$$\rho = \alpha\eta \quad (20)$$

the differential equation (19) transforms into a modified Bessel equation of zero order as

$$\frac{d^2B}{d\rho^2}+\frac{1}{\rho}\frac{dB}{d\rho}-B = 0. \quad (21)$$

The bounded solution is

$$B = CI_0(\rho) \quad (22)$$

where I_0 is the zeroth-order Bessel function of the second kind, and C is a constant of integration.

It must be noted that the solution of equation (22) satisfies the symmetry condition of equation (14), since

$$\frac{dB}{d\eta} = C\alpha I_1(\rho) = 0 \quad \text{when} \quad \rho = 0 \quad (23)$$

where I_1 is the first-order Bessel function of the second kind and vanishes at the origin.

Imposing the boundary condition

$$B(1;\alpha) = \frac{T_1}{\alpha\pi} \quad (24)$$

on equation (22), one obtains

$$C = \frac{T_1}{\alpha\pi}\frac{1}{I_0(\alpha)}. \quad (25)$$

Therefore

$$b_0 = B = \frac{T_1}{\alpha\pi}\frac{I_0(\alpha\eta)}{I_0(\alpha)}. \quad (26)$$

Substituting this into the t expression, equation (9)

$$t(\eta,\xi) = \frac{T_1/2}{\pi/2}\int_0^\infty \frac{I_0(\alpha\eta)}{I_0(\alpha)}\frac{\sin\alpha\xi}{\alpha}d\alpha \quad (27)$$

which is the steady temperature distribution in a circular rod subjected to a step-change in the wall temperature. This expression is numerically integrated using the Runge–Kutta technique for various values of η and ξ, and the resulting temperature distribution curves are presented in ref. [22].

2.2. *Solution of the differential equation for* b_1

The differential equation for b_1 can be written from the second equation of (18) as

$$b_1''+\frac{1}{\eta}b_1-\alpha^2 b_1 = 0. \quad (28)$$

Again, the primes indicate differentiation with respect to the dimensionless radial coordinate η. If the independent variable η is changed to ρ by the transformation given in equation (20), the solution to equation (28) is obtained as

$$b_1 = I_0(\rho) \tag{29}$$

which is the modified Bessel equation of zero order.

2.3. *Solution of the differential equation for* a_1

The differential equation for a_1 can be written from the fifth equation of (18) as

$$a_1''+\frac{1}{\eta}a_1'-\alpha^2 a_1 = \alpha(1-\eta^2)b_0+\frac{T}{\pi}(1-\eta^2). \tag{30}$$

The solution of a_1 for equation (30) is then

$$a_1 = \frac{m}{\alpha^2}\frac{I_0(\rho)}{I_0^2(\alpha)}\{(2)^2\bar{C}_2+(4)^2D_1\bar{C}_4+(6)^2D_2\bar{C}_6+\ldots\}$$
$$-\frac{m}{\alpha^2}\frac{1}{I_0(\alpha)}\{\bar{C}_0+\bar{D}_1\bar{C}_2+\bar{D}_2\bar{C}_4+\bar{D}_4\bar{C}_6+\ldots\} \tag{31}$$

where the expressions for $\bar{C}_0$, $\bar{C}_2$, $\bar{C}_4$, $\bar{C}_6$, $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_4$, m and $I_0(\rho)$ are obtained explicitly and given in the Appendix.

2.4. *The solution of the differential equation for* b_2

The differential equation for b_2 can be written from the third equation of (18) as

$$b_2''+\frac{1}{\eta}b_2'-\alpha^2 b_2 = -\alpha(1-\eta^2)a_1. \tag{32}$$

The solution of b_2 for equation (32) is then

$$b_2 = \frac{m}{\alpha^3}\frac{1}{I_0(\alpha)}\{I_0\bar{\bar{C}}_0+[\alpha^2\bar{I}_2+(2)^2\bar{I}_0]\bar{\bar{C}}_2$$
$$+[\alpha^4\bar{I}_4+\alpha^2(4)^2\bar{I}_2+(4)^2(2)^2\bar{I}_0\bar{\bar{C}}_4]$$
$$+[\alpha^6\bar{I}_6+\alpha^4(6)^2\bar{I}_4+\alpha^2(6)^2(4)^2\bar{I}_2$$
$$+(6)^2(4)^2(2)^2\bar{I}_0]\bar{\bar{C}}_6+\ldots\} \tag{33}$$

where the expressions for $\bar{\bar{C}}_0$, $\bar{\bar{C}}_2$, $\bar{\bar{C}}_4$, $\bar{\bar{C}}_6$, $\bar{I}_0$, $\bar{I}_2$, $\bar{I}_4$, $\bar{I}_6$ and m have been obtained explicitly and are given in the Appendix.

2.5. *The solution of the differential equation for* a_2

The differential equation for a_2 can be written from the sixth equation of (18) as

$$a_2''+\frac{1}{\eta}a_2'-\alpha^2 a_2 = \alpha(1-\eta^2)b_1. \tag{34}$$

Using the same procedure as before, the solution for a_2 from equation (34) is

$$a_2 = \frac{1}{\alpha}\frac{I_0(\rho)}{I_0(\alpha)}\{(2)^2\bar{C}_2+(4)^2D_1\bar{C}_4+(6)^2D_2\bar{C}_6+\ldots\}$$
$$-\frac{1}{\alpha}\{\bar{C}_0+\bar{D}_1\bar{C}_2+\bar{D}_2\bar{C}_4+\bar{D}_4\bar{C}_6+\ldots\} \tag{35}$$

where the expressions for $\bar{C}_0$, $\bar{C}_2$, $\bar{C}_4$, $\bar{C}_6$, $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_4$ have been obtained explicitly and are given in the Appendix.

2.6. *The solution of the differential equation for* b_3

The differential equation for b_3 can be written from the fourth equation of (18) as

$$b_3''+\frac{1}{\eta}b_3'-\alpha^2 b_3 = -\alpha(1-\eta^2)a_2. \tag{36}$$

Again, using the same procedure as outlined before, the solution for b_3 from equation (36) is

$$b_3 = \frac{1}{\alpha^2}\{\bar{I}_0\bar{\bar{C}}_0+[\alpha^2\bar{I}_2+(2)^2\bar{I}_0]\bar{\bar{C}}_2$$
$$+[\alpha^4\bar{I}_4+\alpha^2(4)^2\bar{I}_2+(4)^2(2)^2\bar{I}_0\bar{\bar{C}}_4]$$
$$+[\alpha^6\bar{I}_6+\alpha^4(6)^2\bar{I}_4+\alpha^2(6)^2(4)^2\bar{I}_2$$
$$+(6)^2(4)^2(2)^2\bar{I}_0]\bar{\bar{C}}_6+\ldots\} \tag{37}$$

where the expressions for $\bar{\bar{C}}_0$, $\bar{\bar{C}}_2$, $\bar{\bar{C}}_4$, $\bar{\bar{C}}_6$, $\bar{I}_0$, $\bar{I}_2$, $\bar{I}_4$, $\bar{I}_6$ have already been defined in the Appendix.

2.7. *The solution for the differential equation for* a_3

The differential equation for a_3 can be written from the last equation of (18) as

$$a_3''+\frac{1}{\eta}a_3'-\alpha^2 a_3 = \alpha(1-\eta^2)b_2. \tag{38}$$

Substituting b_2 from equation (33) into equation (38) and following the same procedure as defined earlier, the solution for a_3 is

$$a_3 = \frac{mI_0(\rho)}{\alpha^4 I_0^2(\alpha)}[\bar{\bar{C}}_0+\bar{\bar{D}}_1\bar{\bar{C}}_2+D_2\bar{\bar{C}}_4+D_4\bar{\bar{C}}_6+\ldots]$$
$$-\frac{m}{\alpha^4 I_0(\alpha)}[\bar{\bar{C}}_0+\bar{D}\bar{\bar{C}}_2+\bar{D}_2\bar{\bar{C}}_4+\bar{D}_4\bar{\bar{C}}_6+\ldots] \tag{39}$$

where the expressions for $\bar{\bar{C}}_0$, $\bar{\bar{C}}_2$, $\bar{\bar{C}}_4$, $\bar{\bar{C}}_6$, D_1, D_2, D_4, $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_4$, have been obtained explicitly and are given in the Appendix. The expressions for b_0, b_1, b_2, b_3, a_1, a_2, and a_3 from equations (26), (29), (33), (37), (31), (35), and (39) are substituted in equation (16) for $\bar{B}$ and A. The expressions for $\bar{B}$ and $\bar{A}$ are in turn substituted in the temperature distribution equation (9). The resulting expression is numerically integrated using the Runge–Kutta technique.

3. HEAT FLUXES AND HEAT TRANSFER COEFFICIENTS

The element of heat flux du, measured in the positive direction r, through an elemental area r = constant cylindrical surface, is given as

$$\mathrm{d}u = -LF\eta\,\mathrm{d}\eta\frac{\partial t(\eta,\xi)}{\partial\eta} \tag{40}$$

with

$$t = (Tk)/(LF)$$

where, in the above equation, k is the thermal conductivity, F a reference heat flux, T and t are the dimensional and dimensionless excess temperatures.

The heat gain rate per unit length of pipe is expressed as

$$U = LF\int_0^{\pi}\left(\frac{\partial t(\eta,\xi)}{\partial \eta}\right)_{\eta=1} d\theta. \tag{41}$$

The heat gain rate is taken to be positive when heat flows into the fluid. Thus

$$U = 2\pi LF\left[\frac{\partial t(\eta,\xi)}{\partial \eta}\right]_{\eta=1} \tag{42}$$

where, in equation (42), $\partial t(\eta,\xi)/\partial\eta$ is the differentiation of the temperature distribution, equation (9), with respect to η.

From an energy balance, the mixed mean or bulk temperature T_m at any section is defined by

$$T_m = (\rho/Q)\int_s wt(\eta,\xi)\,ds. \tag{43}$$

A mixed mean excess temperature, E_m, corresponding to equation (43) is obtained from

$$E_m = (T_m - T_{wall}) = \frac{\rho}{Q}\int_s wt(\eta,\xi)\,ds \tag{44}$$

where ρ is the density, Q the mass flow rate, T_{wall} the outer wall temperature of the pipe, and s and ds are the full and elemental cross-sectional areas of pipe, respectively.

After some calculations

$$E_m = -\frac{FL}{K}\frac{J}{I_{00}}.$$

Additionally

$$J = (1/Re)\int_0^1 wt(\eta,\xi)\eta\,d\eta$$

$$w = Re(1-\eta^2)$$

$$I_{00} = 1/4. \tag{45}$$

The mean convective heat transfer coefficient, h, may be defined as

$$U = (T_{wall} - T_m)Ph \tag{46}$$

where

$$P = 2\pi L. \tag{47}$$

The average Nusselt number for the wall based on the hydraulic diameter of the pipe is given as

$$Nu = hD_h/k \tag{48}$$

where D_h is the hydraulic diameter and is given by

$$D_h = 4S/P = 2L \tag{49}$$

and S is given by

$$S = \pi L^2.$$

Equating the two forms of the rates of heat flows in equations (42) and (46) and neglecting the heat generation, the result simplifies to give

$$Nu = \frac{I_{00}}{J}\left[\frac{\partial t(\eta,\xi)}{\partial \eta}\right]_{\eta=1}. \tag{50}$$

4. DISCUSSION OF NUMERICAL RESULTS AND CONCLUSIONS

Convective heat transfer for laminar flow in a circular pipe subject to the constant wall temperature case for various Peclet numbers ($0 < Pe < \infty$) is analyzed, and results are presented in Figs. 2–5. The analytical expressions for a mixed mean temperature, equation (45), constant heat flux, equation (42), and

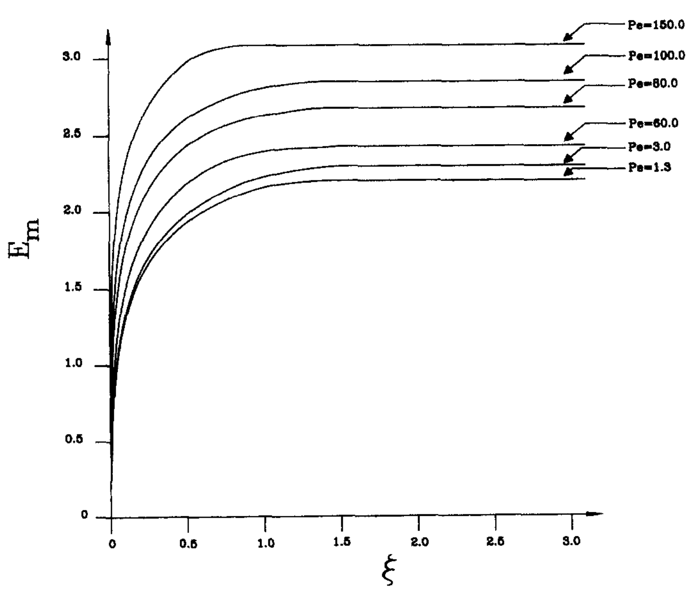


FIG. 2. Mixed mean temperature defined by equation (45) as a function of axial position $\xi = x/R$ for discrete values of Pe.

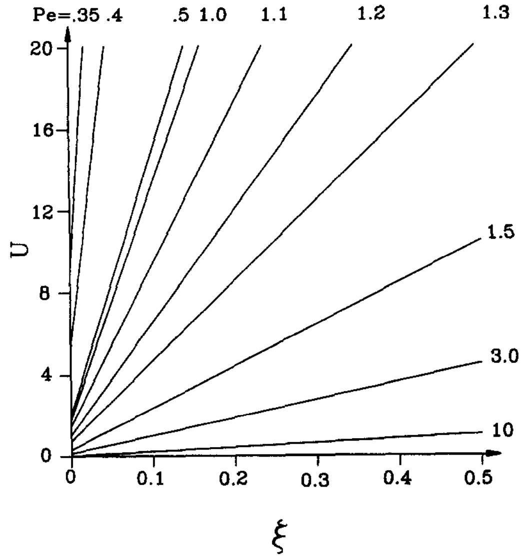


FIG. 3. Heat flux defined by equation (42) as a function of axial position $\xi = x/R$ for small values of Pe.

Nusselt number, equation (50), using the Fourier integral transform, are obtained. These expressions are plotted vs the longitudinal coordinate of the circular pipe (ξ) for various Peclet numbers.

In Fig. 2, equation (45), for the mixed bulk temperature, E_m, is numerically evaluated using the Runge–Kutta method for various values of ξ and Pe. The results obtained are used to plot the temperature with axial heat conduction vs the longitudinal coordinate of a circular pipe (ξ) for Peclet numbers of 1.3, 3.0, 60.0, 80.0, 100.0, 150.0. The curves were compared with those of Colle [18], and rather close agreement was observed. Furthermore, in the case of $\lambda = 0$, the mixed mean bulk temperature (equation (45)) reduces to pure conduction (equation (27)), and the result presented in this figure agrees completely with the one given in Fig. 4 of ref. [22].

In Figs. 3 and 4, equation (42) is used to plot the heat flux vs the longitudinal coordinate axes (ξ) for various Peclet numbers. Figure 3 represents the heat flux for small Peclet numbers (Pe = 0.35, 0.4, 0.5, 1.0, 1.1, 1.2, 1.3, 1.5, 3.0, and 10), and Fig. 4 represents the heat flux for large Peclet numbers (Pe = 30, 50, 60, 80, 100, 150, 200, 300, and 900). Again, the curves presented in Figs. 3 and 4 were compared with the results of Colle [18], and a rather close agreement was observed. Using equation (50), Fig. 5 shows the representative curves for Nusselt numbers with axial heat conduction vs the longitudinal coordinate axes (ξ) for Peclet numbers of 0.5, 1.5, 3.0, 5.0, 10.0, 30.0, 50.0, 100.0, and 1000. Inspection of these curves indicate a uniform asymptotic value of 3.6565 for Nusselt numbers compared to the existing classical value of 3.6568. In addition, the computed asymptotic Nusselt numbers for 17–41 iterations of the Fourier integral transform are given in Table 1.

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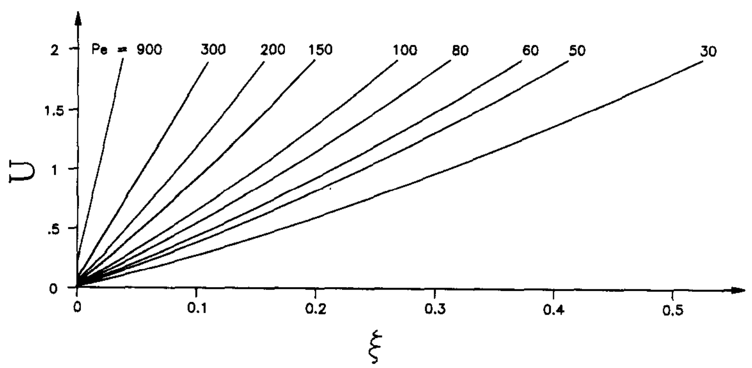

FIG. 4. Heat flux defined by equation (42) as a function of axial position $\xi = x/R$ for large values of Pe.

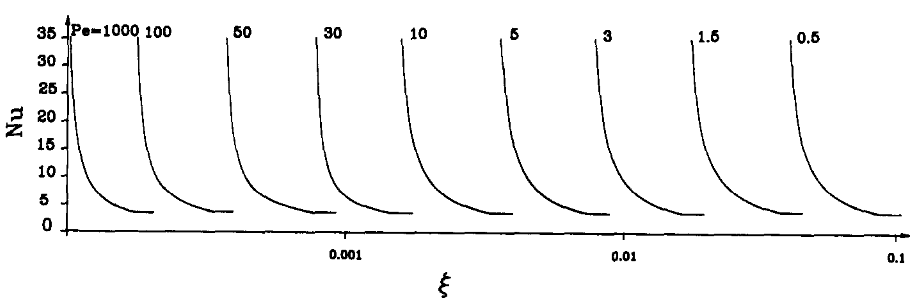

FIG. 5. Nusselt numbers defined by equation (50) as a function of axial position $\xi = x/R$ for discrete values of Pe.

Table 1. The asymptotic Nusselt number for small and large values of Pe by Fourier integral transform, equation (50)

	Pe								
	0.5	1.5	3.0	5	10	50	100	1000	10 000
Nu	3.6785	3.6622	3.6620	3.6597	3.6580	3.6555	3.6555	3.6555	3.6555
No. of iterations	41	35	33	31	29	25	25	17	17

Inspection of the above table indicates using equation (50) will result in slower convergence for Nusselt numbers for small Peclet numbers, compared to large Peclet numbers.

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APPENDIX

$$C_0 = 1, \quad C_2 = \frac{1}{(1)(2)^2},$$
$$C_4 = \frac{1}{(1)^2(2)^2(2)^4}, \quad C_6 = \frac{1}{(1)^2(2)^2(3)^2(2)^6} \quad \text{(A1)}$$

$$m = \frac{T}{\pi} \quad \text{(A2)}$$

$$I_0(\rho) = C_0 + C_2\rho^2 + C_4\rho^4 + C_6\rho^6 + \dots \quad \text{(A3)}$$

$$\bar{C}_0 = C_0, \quad \bar{C}_2 = C_2 - \frac{C_0}{\alpha^2},$$
$$\bar{C}_4 = C_4 - \frac{C_2}{\alpha^2}, \quad \bar{C}_6 = C_6 - \frac{C_4}{\alpha^2} \quad \text{(A4)}$$

$$D_1 = \alpha^2 + (2)^2, \quad D_2 = \alpha^4 + (4)^2\alpha^2 + (4)^2(2)^2,$$
$$D_4 = \alpha^6 + \alpha^4(6)^2 + (\alpha)^2(6)^2(2)^2 + (6)^2(4)^2(2)^2 \quad \text{(A5)}$$

$$E_0 = \frac{C_0}{I_0(\alpha)}, \quad E_2 = \frac{C_2}{I_0(\alpha)}, \quad E_4 = \frac{C_4}{I_0(\alpha)}, \quad E_6 = \frac{C_6}{I_0(\alpha)} \quad \text{(A6)}$$

$$\bar{D}_1 = \rho^2 + (2)^2, \quad \bar{D}_2 = \zeta^4 + (4)^2\rho^6 + (4)^2(2)^2,$$
$$\bar{D}_4 = \rho^6 + (6)^2\rho^4 + (6)^2(4)^2\rho^2 + (6)^2(4)^2(2)^2 \quad \text{(A7)}$$

$$\bar{\bar{C}}_0 = -C_0 + (E_0 - 1)(2)^2\bar{C}_2 + (D_1E_0 - (2)^2)(4)^2\bar{C}_4$$
$$+ [D_2E_0 - (4)^2(2)^2](6)^2\bar{C}_6 \quad \text{(A8)}$$

$$\bar{\bar{C}}_2 = \bar{C}_2\left[(2)^2E_2 - 1 + \frac{1}{\alpha^2}(2)^2(1 - E_0)\right]$$
$$+ \bar{C}_4\left\{D_1E_2 - 1 + \frac{1}{\alpha^2}[(2)^2 - D_1E_0]\right\}(4)^2$$
$$+ \bar{C}_6\left\{D_2E_0 - (4)^2 + \frac{1}{\alpha^2}[(4)^2(2)^2\right.$$

$$-D_2E_0]\Big\}(6)^2+\frac{C_0}{\alpha^2} \qquad \text{(A9)}$$

$$\bar{\bar{C}}_4=\bar{C}_2\left[(2)^2E_4+\frac{1}{\alpha^2}(1-(2)^2E_2)\right]$$
$$+\bar{C}_4\left\{(4)^2D_1E_4-1+\frac{1}{\alpha^2}(4)^2[1-D_1E_2]\right\}$$
$$+\bar{C}_6\left[(6)^2(D_2E_4-1)-\frac{1}{\alpha^2}(6)^2(D_2E_2-(4)^2)\right] \qquad \text{(A10)}$$

$$\bar{\bar{C}}_6=\frac{1}{\alpha^2}(2)^2\bar{C}_2\bar{C}_6+\bar{C}_4\Big\{(4)^2D_1E_6$$
$$-\frac{1}{\alpha^2}[(4)^2D_1E_4-1]\Big\}+\bar{C}_6[(6)^2D_2E_6]$$
$$-1-\frac{1}{\alpha^2}(6)^2[D_2E_4-1] \qquad \text{(A11)}$$

$$\bar{I}_0=1-\frac{I_0(\rho)}{I_0(\alpha)},\quad \bar{I}_2=\frac{\rho^2}{\alpha^2}-\frac{I_0(\rho)}{I_0(\alpha)},$$
$$\bar{I}_4=\frac{\rho^4}{\alpha^4}-\frac{I_0(\rho)}{I_0(\alpha)},\quad \bar{I}_6=\frac{\rho^6}{\alpha^6}-\frac{I_0(\rho)}{I_0(\alpha)}. \qquad \text{(A12)}$$

SOLUTION EXACTE DU PROBLEME DE GRAETZ AVEC CONDUCTION AXIALE DE CHALEUR

Résumé—La convection thermique d'un écoulement visqueux hydrodynamiquement établi dans un tube circulaire est analysée sans hypothèse simplificatrice telle que faible nombre de Reynolds ou de Péclet. Le tube est soumis à un changement d'échelon de température de paroi. Une approche utilisant la technique de la transformation de Fourier permet d'obtenir analytiquement les expressions de la distribution de la température, du flux thermique et du nombre de Nusselt. Les effets de la conduction axiale sont inclus dans ces expressions, à la fois pour les directions amont ou aval, pour des nombres de Péclet entre 0 et ∞. En prenant la transformée de Fourier de la température et en développant les coefficients de la transformée de la température en fonction du nombre de Péclet, l'équation de l'énergie, avec température de paroi discontinue et conduction thermique longitudinale, est transformée en un système d'équations différentielles, lesquelles sont résolues successivement. Des courbes représentatives, sont données pour illustrer la variation de la température moyenne, du flux thermique et du nombre de Nusselt avec des paramètres pertinents. Le nombre de Nusselt asymptotiques pour les petits ou les grands nombres de Péclet est obtenu égal à 3,6565 au lieu de la valeur classique de 3,6568. On discute aussi la signification de chaque courbe.

EINE GENAUE LÖSUNG DES ERWEITERTEN GRAETZ-PROBLEMS MIT AXIALER WÄRMELEITUNG

Zusammenfassung—Die konvektiven Wärmeübertragungs-Eigenschaften einer hydrodynamisch voll ausgebildeten viskosen Strömung in einem kreisrunden Rohr werden ohne vereinfachende Annahmen, wie niedrige Reynolds- oder Peclet-Zahlen, analysiert. Hierbei wird das Rohr einer sprunghaften Änderung der Wandtemperatur ausgesetzt. Eine direkte Näherung mit Hilfe der Technik der Fourier-Transformation wird verwendet, um analytische Ausdrücke für Temperaturverteilung, Wärmestrom und Nusselt-Zahlen zu erhalten. Die Effekte der axialen Wärmeleitung gehen in diese Ausdrücke mit ein, sowohl in aufwärts- als auch abwärtsgerichteten Strömungen für Peclet-Zahlen zwischen null und unendlich. Der Energiesatz mit unstetiger Wandtemperatur und längsgerichteter Wärmeleitung wird in eine Schar von gewöhnlichen Differentialgleichungen überführt, wobei zuerst die Fourier-Transformation des Temperaturfeldes genommen wird und die Koeffizienten der umgewandelten Temperatur in Abhängigkeit von der Peclet-Zahl angegeben werden. Die so erhaltene Schar von Gleichungen wird daraufhin sukzessive gelöst. Die typischen Kurven, die die Änderung von Gesamt-Temperatur, Wärmestrom und Nusselt-Zahlen mit den dazugehörigen Parametern zeigen, sind grafisch dargestellt. Die asymptotische Nusselt-Zahl ergibt sich für kleine ebenso wie für große Peclet-Zahlen zu 3,6565, während der genaue klassische Wert 3,6568 beträgt. Die Besonderheit jeder einzelnen Kurve wird ebenfalls diskutiert.

ТОЧНОЕ РЕШЕНИЕ ОБОБЩЕННОЙ ЗАДАЧИ ГРЕТЦА С УЧЕТОМ АКСИАЛЬНОЙ ТЕПЛОПРОВОДНОСТИ

Аннотация—Без привлечения таких упрощенных предположений, как низкие числа Рейнольдса или Пекле, анализируются характеристики конвективного теплопереноса при гидродинамически полностью развитом вязком течении в круглой трубе. Затем производится ступенчатый нагрев стенки трубы. С целью получения аналитических выражений для температурного поля, теплового потока и чисел Нуссельта используется прямой подход, основанный на методе преобразований фурье. В этих выражениях учитываются аксиальная теплопроводность как вверх, так и вниз по потоку при числах Пекле, изменяющихся от 0 до ∞. Применяя Фурье-преобразование к температурному полю и разлагая в ряд по числам Пекле коэффициенты преобразованной температуры, уравнение энергии со ступенчатой температурой стенки и продольной теплопроводностью преобразуется в систему обыкновенных дифференциальных уравнений, которая затем решается. Построены характерные кривые, иллюстрирующие зависимость среднемассовой температуры, теплового потока и значений числа Нуссельта от соответствующих параметров. Асимптотическое значение числа Нуссельта, полученное как при малых, так и при больших числах Пекле, составило 3.6565. Точное классическое значение равняется 3.6568. Анализируется также характер каждой кривой.